

Compact complete null curves in Complex 3-space

Antonio Alarcón and Francisco J. López

Abstract We prove that for any open orientable surface S of finite topology, there exist a Riemann surface \mathcal{M} , a relatively compact domain $M \subset \mathcal{M}$ and a continuous map $X : \overline{\mathcal{M}} \rightarrow \mathbb{C}^3$ such that:

- \mathcal{M} and M are homeomorphic to S , $\mathcal{M} - M$ and $\mathcal{M} - \overline{M}$ contain no relatively compact components in \mathcal{M} ,
- $X|_M$ is a complete null holomorphic curve, $X|_{\overline{\mathcal{M}} - M} : \overline{\mathcal{M}} - M \rightarrow \mathbb{C}^3$ is an embedding and the Hausdorff dimension of $X(\overline{\mathcal{M}} - M)$ is 1.

Moreover, for any $\epsilon > 0$ and compact null holomorphic curve $Y : N \rightarrow \mathbb{C}^3$ with non-empty boundary $Y(\partial N)$, there exist Riemann surfaces M and \mathcal{M} homeomorphic to N° and a map $X : \overline{\mathcal{M}} \rightarrow \mathbb{C}^3$ in the above conditions such that $\delta^H(Y(\partial N), X(\overline{\mathcal{M}} - M)) < \epsilon$, where $\delta^H(\cdot, \cdot)$ means Hausdorff distance in \mathbb{C}^3 .

Keywords Null holomorphic curves, Calabi-Yau problem, Plateau problem.

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1. Introduction

Given an open Riemann surface \mathcal{N} , a null (holomorphic) curve in \mathbb{C}^3 is a holomorphic immersion $X = (X_j)_{j=1,2,3} : \mathcal{N} \rightarrow \mathbb{C}^3$ satisfying that $\sum_{j=1}^3 (dX_j)^2 = 0$, where d is the complex differential. In this paper we deal with the existence of compact complete null curves in \mathbb{C}^3 accordingly to the following

Definition 1.1. *Let M be a relatively compact domain in an open Riemann surface. A continuous map $X : \overline{\mathcal{M}} \rightarrow \mathbb{C}^3$ is said to be a compact complete null curve in \mathbb{C}^3 if $X|_M$ is a complete null curve.*

The first approach to this matter was made by Martín and Nadirashvili in the context of simply connected minimal surfaces in \mathbb{R}^3 [MN]. The corresponding finite topology case was considered in [Al], see also [Na2, AN] for other related questions. Compact complete minimal surfaces manifest the interplay between two well studied topics on minimal surface theory: Plateau's and Calabi-Yau's problems. The first one consists of finding a compact minimal surface spanning a given closed curve in \mathbb{R}^3 , and it was independently solved by Douglas [Do] and Radó [Ra]. The second one deals with the construction of complete minimal surfaces in \mathbb{R}^3 with bounded coordinate functions, and the most relevant examples were given by Jorge-Xavier [JX] and Nadirashvili [Na1].

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In the very last few years, the study of the Calabi-Yau problem has evolved in the direction of null curves in \mathbb{C}^3 . Observe that a holomorphic map $X : \mathcal{N} \rightarrow \mathbb{C}^3$ is a null curve if and only if both $\Re(X) : \mathcal{N} \rightarrow \mathbb{R}^3$ and $\Im(X) : \mathcal{N} \rightarrow \mathbb{R}^3$ are conformal minimal immersions. Furthermore, the Riemannian metric on \mathcal{N} induced by X is twice the one induced by $\Re(X)$ and $\Im(X)$. Therefore, complete bounded null curves in \mathbb{C}^3 provide complete bounded minimal surfaces in \mathbb{R}^3 with well defined and bounded conjugate surface, and viceversa. Existence of a vast family of complete bounded null curves with arbitrary topology and other additional properties is known [MUY1, AL2].

If M is a relatively compact domain in an open Riemann surface and $F : \overline{M} \rightarrow \mathbb{R}^3$ is a continuous map such that $F|_M : M \rightarrow \mathbb{R}^3$ is a conformal complete minimal immersion whose conjugate immersion $(F|_M)^* : M \rightarrow \mathbb{R}^3$ is well defined, then $(F|_M)^*$ does not necessarily extend as a continuous map to \overline{M} . Therefore, it is natural to wonder whether there exist compact complete null curves in \mathbb{C}^3 .

In this paper we answer this question, proving considerably more.

Main Theorem. *Let S be an open orientable surface of finite topology.*

Then there exist a Riemann surface \mathcal{M} , a relatively compact domain $M \subset \mathcal{M}$ and a continuous map $X : \overline{M} \rightarrow \mathbb{C}^3$ such that:

- *\mathcal{M} and M are homeomorphic to S , $\mathcal{M} - M$ and $\mathcal{M} - \overline{M}$ contain no relatively compact components in \mathcal{M} ,*
- *$X|_M$ is a complete null curve, $X|_{\overline{M}-M} : \overline{M} - M \rightarrow \mathbb{C}^3$ is an embedding and the Hausdorff dimension of $X(\overline{M} - M)$ is 1.*

Moreover, for any $\epsilon > 0$ and for any compact null curve $Y : N \rightarrow \mathbb{C}^3$ with non-empty boundary $Y(\partial N)$, there exist Riemann surfaces M and \mathcal{M} homeomorphic to N° and a map $X : \overline{M} \rightarrow \mathbb{C}^3$ in the above conditions such that $\delta^H(Y(\partial N), X(\overline{M} - M)) < \epsilon$, where $\delta^H(\cdot, \cdot)$ means Hausdorff distance in \mathbb{C}^3 .

Unfortunately, the techniques we use do not give enough control over the topology of $\overline{M} - M$ to assert that it consists, for instance, of a finite collection of Jordan curves. Concerning the second part of the theorem, it is worth mentioning that there exist Jordan curves in \mathbb{C}^3 which are not spanned by any null curve. Main Theorem actually follows from a more general density result (see Theorem 5.1). Some other similar existence results for complex curves in \mathbb{C}^2 , null holomorphic curves in the complex Lie group $\mathrm{SL}(2, \mathbb{C})$, Bryant surfaces in hyperbolic 3-space \mathbb{H}^3 and minimal surfaces in \mathbb{R}^3 can be also derived from it, see Corollary 5.3. See [MUY1, MUY2, AL2] for related results.

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2. Preliminaries

We denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and metric in \mathbb{K}^n , where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and for any compact topological space K and continuous map $f : K \rightarrow \mathbb{K}^n$ we denote by $\|f\| = \max\{\|f(p)\| \mid p \in K\}$ the maximum norm of f on K . We also denote by $\iota = \sqrt{-1}$.

2.1. Riemann surfaces

Given a Riemann surface \mathcal{M} with non empty boundary, we denote by $\partial\mathcal{M}$ the (possibly non-connected) 1-dimensional topological manifold determined by its boundary points. For

any $A \subset \mathcal{M}$, we denote by A° , \overline{A} and $\text{Fr}(A) = \overline{A} - A^\circ$ the interior of A in \mathcal{M} , the closure of A in \mathcal{M} and the topological frontier of A in \mathcal{M} . Open connected subsets of \mathcal{M} will be called *domains*, and those proper topological subspaces of \mathcal{M} being Riemann surfaces with boundary are said to be *regions*.

A Riemann surface \mathcal{M} is said to be a *bordered Riemann surface* if it is compact, $\partial\mathcal{M} \neq \emptyset$ and $\partial\mathcal{M}$ is smooth. A compact region of \mathcal{M} is said to be a *bordered region* if it is a bordered Riemann surface.

Remark 2.1. Throughout this paper, \mathcal{R}_0 and ω will denote a fixed but arbitrary bordered Riemann surface and a smooth conformal metric on \mathcal{R}_0 , respectively. We call \mathcal{R} the open Riemann surface $\mathcal{R}_0 - \partial\mathcal{R}_0$, and write $e \in \mathbb{N}$ for the number of ends of \mathcal{R} (or equivalently, for the number of connected components of $\partial\mathcal{R}_0$).

A subset $A \subset \mathcal{R}$ is said to be *Runge* if $j_* : \mathcal{H}_1(A, \mathbb{Z}) \rightarrow \mathcal{H}_1(\mathcal{R}_0, \mathbb{Z})$ is injective, where $j : A \rightarrow \mathcal{R}_0$ is the inclusion map. If A is a compact region in \mathcal{R} , this simply means that $\mathcal{R} - A$ has no relatively compact components in \mathcal{R} .

In the remainder of this subsection we introduce the necessary notations for a precise statement of the approximation result by null curves in Lemma 2.12, which is the starting point of the paper.

A 1-form θ on a subset $S \subset \mathcal{R}$ is said to be of type $(1, 0)$ if for any conformal chart (U, z) in \mathcal{R} , $\theta|_{U \cap S} = h(z)dz$ for some function $h : U \cap S \rightarrow \mathbb{C}$.

Definition 2.2. Assume that $S \subset \mathcal{R}$ is compact. We say that

- a function $f : S \rightarrow \mathbb{C}$ can be uniformly approximated on S by holomorphic functions in \mathcal{R} if and only if there exists a sequence of holomorphic functions $\{f_n : \mathcal{R} \rightarrow \mathbb{C}\}_{n \in \mathbb{N}}$ such that $\{f_n - f\}_{n \in \mathbb{N}} \rightarrow 0$ uniformly on S , and
- a 1-form θ on S can be uniformly approximated on S by holomorphic 1-forms in \mathcal{R} if and only if there exists a sequence of holomorphic 1-forms $\{\theta_n\}_{n \in \mathbb{N}}$ on \mathcal{R} such that $\{\frac{\theta_n - \theta}{dz}\}_{n \in \mathbb{N}} \rightarrow 0$ uniformly on $S \cap U$, for any closed conformal disc (U, dz) on \mathcal{R} .

The following definition is crucial in our arguments.

Definition 2.3 (Admissible set). A compact subset $S \subset \mathcal{R}$ is said to be *admissible* if and only if

- S is Runge,
- $M_S := \overline{S}^\circ$ is a finite collection of pairwise disjoint compact regions in \mathcal{R} with \mathcal{C}^0 boundary,
- $C_S := \overline{S} - M_S$ consists of a finite collection of pairwise disjoint analytical Jordan arcs, and
- any component α of C_S with an endpoint $P \in M_S$ admits an analytical extension β in \mathcal{R} such that the unique component of $\beta - \alpha$ with endpoint P lies in M_S .

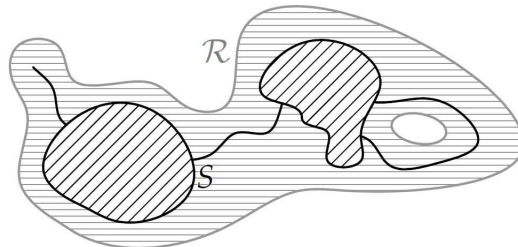


FIGURE 2.1. An admissible set S in \mathcal{R} .

The next one deals with the notion of smoothness of functions and 1-forms on admissible subsets.

Definition 2.4. Assume that S is an admissible subset of \mathcal{R} .

- A function $f : S \rightarrow \mathbb{C}$ is said to be smooth if and only if $f|_{M_S}$ admits a smooth extension f_0 to an open domain V in \mathcal{R} containing M_S , and for any component α of C_S and any open analytical Jordan arc β in \mathcal{R} containing α , f admits a smooth extension f_β to β satisfying that $f_\beta|_{V \cap \beta} = f_0|_{V \cap \beta}$.
- A 1-form θ on S is said to be smooth if and only if $(\theta|_{S \cap U})/dz$ is a smooth function for any closed conformal disk (U, z) on \mathcal{R} such that $S \cap U$ is an admissible set.

Given a smooth function $f : S \rightarrow \mathbb{C}$ on an admissible $S \subset \mathcal{R}$, we denote by df the 1-form of type (1,0) given by $df|_{M_S} = d(f|_{M_S})$ and $df|_{\alpha \cap U} = (f \circ \alpha)'(x)dz|_{\alpha \cap U}$, where $(U, z = x + iy)$ is any conformal chart on \mathcal{R} satisfying that $z(\alpha \cap U) \subset \mathbb{R}$. Then df is well defined and smooth. The \mathcal{C}^1 -norm of f on S is given by

$$\|f\|_1 = \max_S \{\|f\| + \|df/\omega\|\}.$$

A sequence of smooth functions $\{f_n\}_{n \in \mathbb{N}}$ on S is said to converge in the \mathcal{C}^1 -topology to a smooth function f on S if $\{\|f - f_n\|_1\}_{n \in \mathbb{N}} \rightarrow 0$. If in addition f_n is (the restriction to S of) a holomorphic function on \mathcal{R} for all n , we also say that f can be uniformly \mathcal{C}^1 -approximated on S by holomorphic functions on \mathcal{R} .

Likewise one can define the notions of smoothness, (vectorial) differential, \mathcal{C}^1 -norm and uniform \mathcal{C}^1 -approximation for maps $f : S \rightarrow \mathbb{C}^k$, $k \in \mathbb{N}$, $S \subset \mathcal{R}$ admissible.

Let S be an admissible subset of \mathcal{R} , and let W be a Runge open subset of \mathcal{R} whose closure \overline{W} in \mathcal{R}_0 is a compact region and $j_* : \mathcal{H}_1(W, \mathbb{Z}) \rightarrow \mathcal{H}_1(\overline{W}, \mathbb{Z})$ is an isomorphism, where $j : W \rightarrow \overline{W}$ is the inclusion map. W is said to be a *tubular neighborhood* of S if $S \subset W$, $(j_0)_* : \mathcal{H}_1(S, \mathbb{Z}) \rightarrow \mathcal{H}_1(\overline{W}, \mathbb{Z})$ is an isomorphism and $\chi(W - S) = \chi(\overline{W} - S) = 0$, where $j_0 : S \rightarrow \overline{W}$ is the inclusion map and $\chi(\cdot)$ means Euler characteristic. In particular, $W - S$ (respectively, $\overline{W} - S^\circ \subset \mathcal{R}_0$) consists of a family of pairwise disjoint open (respectively, compact) annuli.

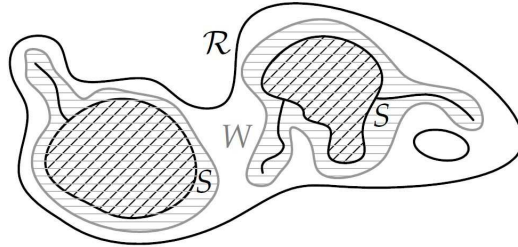


FIGURE 2.2. A tubular neighborhood W of an admissible set S in \mathcal{R} .

For instance, assume that S is a finite family of pairwise disjoint smooth Jordan curves γ_j , $j = 1, \dots, k$, in \mathcal{R} , take $\epsilon > 0$ and set $W = \{p \in \mathcal{R} \mid \text{dist}_\omega(p, S) < \epsilon\}$, where dist_ω means Riemannian distance in (\mathcal{R}_0, ω) . If ϵ is small enough, the exponential map $F : S \times [-\epsilon, \epsilon] \rightarrow \overline{W}$, $F(p, t) = \exp_p(tn(p))$, is a diffeomorphism and $W = F(S \times (-\epsilon, \epsilon))$, where n is a normal field along S in (\mathcal{R}, ω) . In this case, W is said to be a *metric tubular neighborhood* of S (of radius ϵ). Furthermore, if $\pi_1 : S \times (-\epsilon, \epsilon) \rightarrow S$ denotes the projection $\pi_1(p, t) = p$, we denote by $\mathfrak{P} : W \rightarrow S$, $\mathfrak{P}(q) := \pi_1(F^{-1}(q))$, the natural orthogonal projection.

Definition 2.5. We denote by $B(\mathcal{R})$ the family of Runge bordered regions $M \subset \mathcal{R}$ such that \mathcal{R} is a tubular neighborhood of M . Given $M, N \in B(\mathcal{R})$, we say that $M < N$ if and only if $M \subset N^\circ$.

2.2. Null curves in \mathbb{C}^3

Throughout this paper we adopt column notation for both vectors and matrices of linear transformations in \mathbb{C}^3 , and make the convention

$$\mathbb{C}^3 \ni (z_1, z_2, z_3)^T \equiv (\Re(z_1), \Im(z_1), \Re(z_2), \Im(z_2), \Re(z_3), \Im(z_3))^T \in \mathbb{R}^6 \quad ((\cdot)^T \text{ means "transpose"}).$$

Let us start this subsection by introducing some operators which are strongly related to the geometry of \mathbb{C}^3 and null curves.

Definition 2.6. Let $A \subset \mathbb{C}^3$. We denote by

- $\langle\langle \cdot, \cdot \rangle\rangle: \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathbb{C}$, $\langle\langle u, v \rangle\rangle = \bar{u}^T \cdot v$, the usual Hermitian inner product in \mathbb{C}^3 ,
- $\langle\langle A \rangle\rangle^\perp = \{v \in \mathbb{C}^3 \mid \langle\langle u, v \rangle\rangle = 0 \forall u \in A\}$,
- $\langle \cdot, \cdot \rangle = \Re(\langle\langle \cdot, \cdot \rangle\rangle)$ the Euclidean scalar product of $\mathbb{C}^3 \equiv \mathbb{R}^6$,
- $\langle A \rangle^\perp = \{v \in \mathbb{C}^3 \mid \langle u, v \rangle = 0 \forall u \in A\}$,
- $\prec \cdot, \cdot \succ: \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathbb{C}$, $\prec u, v \succ = u^T \cdot v$, and
- $\prec A \succ^\perp = \{v \in \mathbb{C}^3 \mid \prec u, v \succ = 0 \forall u \in A\}$.

Notice that $\prec \bar{u} \succ^\perp = \langle\langle u \rangle\rangle^\perp \subset \langle u \rangle^\perp$ for all $u \in \mathbb{C}^3$, and the equality holds if and only if $u = 0 \in \mathbb{C}^3$.

Definition 2.7. A vector $u \in \mathbb{C}^3 - \{0\}$ is said to be null if and only if $\prec u, u \succ = 0$. We denote by $\Theta = \{u \in \mathbb{C}^3 - \{0\} \mid u \text{ is null}\}$.

A basis $\{u_1, u_2, u_3\}$ of \mathbb{C}^3 is said to be $\prec \cdot, \cdot \succ$ -conjugate if and only if $\prec u_j, u_k \succ = \delta_{jk}$, $j, k \in \{1, 2, 3\}$.

We denote by $\mathcal{O}(3, \mathbb{C})$ the complex orthogonal group $\{A \in \mathcal{M}_3(\mathbb{C}) \mid A^T \cdot A = I_3\}$, or in other words, the group of matrices whose column vectors determine a $\prec \cdot, \cdot \succ$ -conjugate basis of \mathbb{C}^3 . We also denote by $A: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ the complex linear transformation induced by $A \in \mathcal{O}(3, \mathbb{C})$.

It is clear that $A(\Theta) = \Theta$ for all $A \in \mathcal{O}(3, \mathbb{C})$.

Let \mathcal{N} be an open Riemann surface.

Definition 2.8. A holomorphic map $X: \mathcal{N} \rightarrow \mathbb{C}^3$ is said to be a null curve if and only if $\prec dX, dX \succ = 0$ and $\langle\langle dX, dX \rangle\rangle$ never vanishes on \mathcal{N} .

Conversely, given an exact holomorphic vectorial 1-form Φ on \mathcal{N} satisfying that $\prec \Phi, \Phi \succ = 0$ and $\langle\langle \Phi, \Phi \rangle\rangle$ never vanishes on \mathcal{N} , then the map $X: \mathcal{N} \rightarrow \mathbb{C}^3$, $X(P) = \int^P \Phi$, defines a null curve in \mathbb{C}^3 .

If $X: \mathcal{N} \rightarrow \mathbb{C}^3$ is a null curve then the pull back metric $X^*(\langle \cdot, \cdot \rangle)$ on \mathcal{N} is determined by the expresion $\langle dX, dX \rangle = \langle\langle dX, dX \rangle\rangle$. Given two subsets $V_1, V_2 \subset \mathcal{N}$, we denote by $\text{dist}_{(\mathcal{N}, X)}(V_1, V_2)$ the distance between V_1 and V_2 in the Riemannian surface $(\mathcal{N}, X^*(\langle \cdot, \cdot \rangle))$.

Remark 2.9. Let $X: \mathcal{N} \rightarrow \mathbb{C}^3$ be a null curve and $A = (a_{jk})_{j,k=1,2,3} \in \mathcal{O}(3, \mathbb{C})$. Then $A \circ X: \mathcal{N} \rightarrow \mathbb{C}^3$ is a null curve as well and $X^*(\langle \cdot, \cdot \rangle) \geq \frac{1}{\|A\|^2} (A \circ X)^*(\langle \cdot, \cdot \rangle)$, where $\|A\| = (\sum_{j,k} |a_{jk}|^2)^{1/2}$.

Definition 2.10. Given a subset $S \subset \mathcal{R}$ we denote by $N(S)$ the space of maps $X: S \rightarrow \mathbb{C}^3$ extending as a null curve to an open neighborhood of S in \mathcal{R} .

Definition 2.11. Let $S \subset \mathcal{R}$ be an admissible subset. A smooth map $X: S \rightarrow \mathbb{C}^3$ is said to be a generalized null curve in \mathbb{C}^3 if and only if $X|_{M_S} \in N(M_S)$, $\prec dX, dX \succ = 0$ on S and $\langle\langle dX, dX \rangle\rangle$ never vanishes on S .

The following technical lemma is a key tool in this paper.

Lemma 2.12 (Approximation Lemma [AL1, AL2]). Let $S \subset \mathcal{R}$ be a connected admissible set and let $X = (X_j)_{j=1,2,3}: S \rightarrow \mathbb{C}^3$ be a generalized null curve.

Then X can be uniformly C^1 -approximated on S by a sequence of null curves $\{Y_n = (Y_{j,n})_{j=1,2,3}: \mathcal{R} \rightarrow \mathbb{C}^3\}_{n \in \mathbb{N}}$. In addition, we can choose $Y_{3,n} = X_3 \forall n \in \mathbb{N}$ provided that X_3 extends holomorphically to \mathcal{R} and dX_3 never vanishes on C_S .

3. Completeness Lemma

In this section we state and prove the technical result which is the kernel of this paper (Lemma 3.1 below). Its proof requires of the approximation result by null curves in Lemma 2.12. Lemma 3.1 has more strength than similar results in [MN, AI], and its proof presents some differences.

Lemma 3.1. *Let $M, N \in \mathcal{B}(\mathcal{R})$ with $N < M$, let \mathcal{T} be a metric tubular neighborhood of ∂M in \mathcal{R} and denote by $\mathfrak{P} : \mathcal{T} \rightarrow \partial M$ the orthogonal projection. Consider $X \in \mathcal{N}(M)$, an analytical map $\mathfrak{F} : \partial M \rightarrow \mathbb{C}^3$, and $\mu > 0$ so that*

$$(3.1) \quad \|X - \mathfrak{F}\| < \mu \quad \text{on } \partial M.$$

Then, for any $\rho > 0$ and $\epsilon > 0$, there exist $\mathcal{M} \in \mathcal{B}(\mathcal{R})$ and $\mathcal{X} \in \mathcal{N}(\mathcal{M})$ such that

- (3.1.a) $N < \mathcal{M} < M$ and $\partial \mathcal{M} \subset \mathcal{T}$,
- (3.1.b) $\mathcal{X}|_{\partial \mathcal{M}} : \partial \mathcal{M} \rightarrow \mathbb{C}^3$ is an embedding,
- (3.1.c) $\rho < \text{dist}_{(\mathcal{M}, \mathcal{X})}(N, \partial \mathcal{M})$,
- (3.1.d) $\|\mathcal{X} - X\|_1 < \epsilon$ on N , and
- (3.1.e) $\|\mathcal{X} - \mathfrak{F} \circ \mathfrak{P}\| < \sqrt{4\rho^2 + \mu^2} + \epsilon$ on $\partial \mathcal{M}$.

Roughly speaking, the above Lemma asserts that a compact null curve $X(M)$ in \mathbb{C}^3 can be perturbed into another compact null curve $\mathcal{X}(\mathcal{M})$ with larger intrinsic radius. This deformation hardly modifies the null curve in a prescribed compact set and, in addition, the effect of the deformation in the boundaries of the null curves is quite controlled. The bounds μ , ρ and $\sqrt{4\rho^2 + \mu^2} + \epsilon$ in (3.1) and Items (3.1.c) and (3.1.e) follow the spirit of Nadirashvili's original construction [Na1] (see (4.1) below).

The null curve \mathcal{X} which proves the lemma will be obtained from X after two different perturbation processes. In the first one, which is enclosed in Claim 3.2 below, the effect of the deformation is strong over a family of Jordan arcs in M (see the definition of the arcs $r_{i,j}$ in Items (C.1), (C.2) and (C.3)) and slight on a compact region containing N . In the second stage, see Claim 3.3, the deformation mainly acts on a family of compact discs (see the definition of $K_{i,j}$ in (E.2)) and hardly works on a compact region containing N .

3.1. Proof of Lemma 3.1

Take $\epsilon_0 > 0$ to be specified later.

By (3.1) and a continuity argument, for any $P \in \partial M$ there exists a simply connected open neighborhood V_P of P in $M \cap \mathcal{T}$ such that

- (A.1) $\mathfrak{P}(Q) \in V_P \cap \partial M, \forall Q \in V_P$,
- (A.2) $\max \{ \|X(Q_1) - X(Q_2)\|, \|\mathfrak{F}(\mathfrak{P}(Q_1)) - \mathfrak{F}(\mathfrak{P}(Q_2))\| \} < \epsilon_0, \forall \{Q_1, Q_2\} \subset V_P$, and
- (A.3) $\|X - \mathfrak{F} \circ \mathfrak{P}\| < \mu$ on V_P .

Set $\mathcal{V} = \{V_P\}_{P \in \partial M}$, and observe that $\mathcal{V} \cap \partial M := \{V_P \cap \partial M\}_{P \in \partial M}$ is an open covering of ∂M . Take $M_0 \in \mathcal{B}(\mathcal{R})$ satisfying that

$$(3.2) \quad N < M_0 < M \quad \text{and} \quad M - M_0^\circ \subset \bigcup_{P \in \partial M} V_P,$$

and note that $\mathcal{V} \cap (M - M_0^\circ) := \{V_P \cap (M - M_0^\circ)\}_{P \in \partial M}$ is an open covering of $M - M_0^\circ$ as well. One has that $M - M_0^\circ = \bigcup_{i=1}^e A_i$, where $\{A_i\}_{i=1}^e$ are pairwise disjoint compact annuli. Write $\alpha_i \subset \partial M_0$ and $\beta_i \subset \partial M$ for the two components of ∂A_i , $i = 1, \dots, e$. For each $m \in \mathbb{N}$, let \mathbb{Z}_m denote the additive cyclic group of integers modulus m . Since $\mathcal{V} \cap (M - M_0^\circ)$ is an open covering of A_i , then there exist $m \in \mathbb{N}$, $m \geq 3$, and a collection of compact Jordan arcs $\{\alpha_{i,j} \mid (i, j) \in \{1, \dots, e\} \times \mathbb{Z}_m\}$ satisfying that

- (B.1) $\cup_{j \in \mathbb{Z}_m} \alpha_{i,j} = \alpha_i$,
 (B.2) $\alpha_{i,j}$ and $\alpha_{i,j+1}$ have a common endpoint $Q_{i,j}$ and are otherwise disjoint for all $j \in \mathbb{Z}_m$, and
 (B.3) $\alpha_{i,j} \cup \alpha_{i,j+1} \subset V_{i,j} \in \mathcal{V}$ for all $j \in \mathbb{Z}_m$,

$\forall i \in \{1, \dots, e\}$.

For any $(i, j) \in \{1, \dots, e\} \times \mathbb{Z}_m$ fix $P_{i,j} \in V_{i,j-1} \cap V_{i,j}$ and $e_{i,j} \in S^5 - \Theta$ such that

$$(3.3) \quad \|\Pi_{i,j}(X(P_{i,j}) - \mathfrak{F}(\mathfrak{P}(P_{i,j})))\| < \epsilon_0,$$

where $\Pi_{i,j} : \mathbb{C}^3 \rightarrow \ll e_{i,j} \gg^\perp$ is the orthogonal projection. This choice is possible since $S^5 - \Theta$ is dense in S^5 . Label

$$w_{i,j} = \frac{\overline{e_{i,j}}}{\prec \overline{e_{i,j}}, e_{i,j} \succ},$$

for all i, j , and notice that

$$(3.4) \quad \ll e_{i,j} \gg^\perp = \prec w_{i,j} \succ^\perp.$$

Since $w_{i,j} \notin \Theta$, then we can take $u_{i,j}, v_{i,j} \in \prec w_{i,j} \succ^\perp$ such that $\{u_{i,j}, v_{i,j}, w_{i,j}\}$ is a $\prec \cdot, \cdot \succ$ -conjugate basis of \mathbb{C}^3 . Denote by $\mathcal{A}_{i,j}$ the complex orthogonal matrix $(u_{i,j}, v_{i,j}, w_{i,j})^{-1}$, for all $i \in \{1, \dots, e\}$ and $j \in \mathbb{Z}_m$.

Let $\{r_{i,j} \mid j \in \mathbb{Z}_m\}$ be a family of pairwise disjoint analytical compact Jordan arcs in A_i such that

- (C.1) $r_{i,j} \subset V_{i,j-1} \cap V_{i,j} \cap V_{i,j+1}$ (see (A.1), (B.2) and (B.3)),
 (C.2) $r_{i,j}$ has initial point $Q_{i,j}$, final point $\mathfrak{P}(Q_{i,j})$ and it is otherwise disjoint from $\alpha_i \cup \beta_i$, and
 (C.3) $S = M_0 \cup (\cup_{i,j} r_{i,j})$ is admissible.

As we announced above, the null curve \mathcal{X} will be obtained from X after two deformation procedures. The first one strongly works in $\cup_{i,j} r_{i,j}$ (see property (D.2) below) and is enclosed in the following

Claim 3.2. *There exists $H \in \mathcal{N}(M)$ such that, for any $(i, j) \in \{1, \dots, e\} \times \mathbb{Z}_m$,*

$$(D.1) \quad \|H - X\| < \epsilon_0 \text{ on } S,$$

(D.2) *if $J \subset r_{i,j}$ is a Borel measurable subset, then*

$$\min\{\mathcal{L}_{\mathbb{C}}((\mathcal{A}_{i,j} \cdot H|_J)_3), \mathcal{L}_{\mathbb{C}}((\mathcal{A}_{i,j+1} \cdot H|_J)_3)\} + \min\{\mathcal{L}_{\mathbb{C}}((\mathcal{A}_{i,j} \cdot H|_{r_{i,j}-J})_3), \mathcal{L}_{\mathbb{C}}((\mathcal{A}_{i,j+1} \cdot H|_{r_{i,j}-J})_3)\} > 2\rho \cdot \max\{\|\mathcal{A}_{i,j}\|, \|\mathcal{A}_{i,j+1}\|\},$$

where $(\cdot)_3$ means third (complex) coordinate and $\mathcal{L}_{\mathbb{C}}(\cdot)$ Euclidean length in \mathbb{C} , and

$$(D.3) \quad \|H - X\|_1 < \epsilon_0 \text{ on } M_0.$$

Proof. Let $r_{i,j}(u)$, $u \in [0, 1]$, be a smooth parameterization of $r_{i,j}$ with $r_{i,j}(0) = Q_{i,j}$ and $r_{i,j}(1) = \mathfrak{P}(Q_{i,j})$. From (A.2) and (C.1) there exists a non-empty open subset $\Xi_{i,j} \subset \mathbb{C}^3$ satisfying that

$$(3.5) \quad X(V_{i,j-1} \cap V_{i,j}) \subset \Xi_{i,j} \quad \text{and} \quad \|w_1 - w_2\| < \epsilon_0 \quad \forall \{w_1, w_2\} \subset \Xi_{i,j}.$$

Then (B.3), (3.5) and (C.1) give that $X(r_{i,j-1} \cup \alpha_{i,j} \cup r_{i,j} \cup \alpha_{i,j+1} \cup r_{i,j+1}) \subset \Xi_{i,j}$. Consider $\lambda_{i,j} \in \Theta$ such that $(\mathcal{A}_{i,j}(\lambda_{i,j}))_3, (\mathcal{A}_{i,j+1}(\lambda_{i,j}))_3 \neq 0$ and

$$(3.6) \quad \{\Lambda_{i,j}(s) := X(Q_{i,j}) + s\lambda_{i,j} \mid s \in [0, 1]\} \subset \Xi_{i,j}.$$

Set $\Lambda_{i,j}^*(s) = \Lambda_{i,j}(1-s)$, $s \in [0, 1]$. Take $n \in \mathbb{N}$ large enough so that

$$(3.7) \quad n \cdot \min\{|(\mathcal{A}_{i,j}(\lambda_{i,j}))_3|, |(\mathcal{A}_{i,j+1}(\lambda_{i,j}))_3|\} > 2\rho \cdot \max\{\|\mathcal{A}_{i,j}\|, \|\mathcal{A}_{i,j+1}\|\}.$$

Set $d_{i,j} : [0, 1] \rightarrow \mathbb{C}^3$,

- $d_{i,j}(u) = \Lambda_{i,j}(nu - b + 1)$ if $u \in [\frac{b-1}{n}, \frac{b}{n}]$ and $b \in \{1, \dots, n\}$ is odd, and

- $d_{i,j}(u) = \Lambda_{i,j}^*(nu - b + 1)$ if $u \in [\frac{b-1}{n}, \frac{b}{n}]$ and $b \in \{1, \dots, n\}$ is even.

Notice that the curves $d_{i,j}$ are continuous, weakly differentiable and satisfy that $\prec d'_{i,j}(u), d'_{i,j}(u) \succ = 0$. Up to replacing $H|_{r_{i,j}}$ for $d_{i,j}$ for all i, j , items (D.1), (D.2) and (D.3) formally hold. To finish, approximate $d_{i,j}$ by a smooth curve $c_{i,j}$ matching smoothly with X at $Q_{i,j}$, and so that the map $\tilde{H} : S \rightarrow \mathbb{C}^3$ given by $\tilde{H}|_M = X$, $\tilde{H}|_{r_{i,j}}(u) = c_{i,j}(u)$ for all $u \in [0, 1]$, i and j , is a generalized null curve satisfying all the above items. Indeed, if $c_{i,j}$ is chosen close enough to $d_{i,j}$, (D.1) follows from (3.6) and (D.2) from (3.7). Apply Lemma 2.12 to \tilde{H} and we are done. \square

Denote by $\Omega_{i,j}$ the closed disc in $M - M_0^\circ$ bounded by $\alpha_{i,j}$, $r_{i,j-1}$, $r_{i,j}$ and a piece, named $\beta_{i,j}$, of β_i connecting $\mathfrak{P}(Q_{i,j-1})$ and $\mathfrak{P}(Q_{i,j})$. Since $V_{i,j}$ is simply connected, then (A.1), (B.3) and (C.1) give that

$$(3.8) \quad \Omega_{i,j} \cup \Omega_{i,j+1} \subset V_{i,j}.$$

Consider simply-connected compact neighborhoods $\tilde{\alpha}_{i,j}$ and $\tilde{r}_{i,j}$ in $M - M_0^\circ$ of $\alpha_{i,j}$ and $r_{i,j}$, respectively, $i = 1, \dots, e$, $j \in \mathbb{Z}_m$, satisfying the following properties:

- (E.1) $\tilde{\alpha}_{i,j} \cap \tilde{r}_{i,a} = \emptyset$, $a \neq j-1, j$, and $\tilde{r}_{i,j} \cap \tilde{r}_{i,a} = \emptyset$, $a \neq j$,
- (E.2) $K_{i,j} := \overline{\Omega_{i,j} - (\tilde{r}_{i,j-1} \cup \tilde{\alpha}_{i,j} \cup \tilde{r}_{i,j})}$ is a compact disc and $K_{i,j} \cap \beta_{i,j}$ is a Jordan arc disjoint from the set $\{\mathfrak{P}(Q_{i,j-1}), \mathfrak{P}(Q_{i,j})\}$ (see Figure 3.1),
- (E.3) $\|H - X\| < \epsilon_0$ on $\overline{M - \cup_{i,j} K_{i,j}}$, and
- (E.4) if $J \subset \tilde{r}_{i,j}$ is an arc connecting $\tilde{\alpha}_{i,j} \cup \tilde{\alpha}_{i,j+1}$ and $\beta_{i,j} \cup \beta_{i,j+1}$, $J_1 = J \cap \Omega_{i,j}$ and $J_2 = J \cap \Omega_{i,j+1}$, then

$$\mathcal{L}_C((\mathcal{A}_{i,j} \cdot H|_{J_1})_3) + \mathcal{L}_C((\mathcal{A}_{i,j+1} \cdot H|_{J_2})_3) > 2\rho \cdot \max\{\|\mathcal{A}_{i,j}\|, \|\mathcal{A}_{i,j+1}\|\}.$$

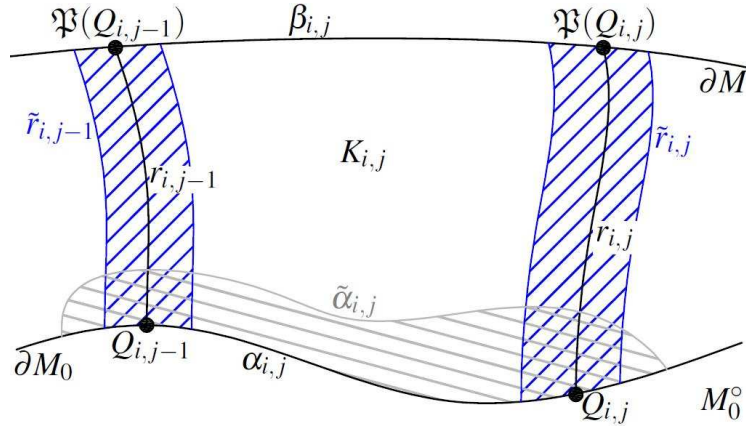


FIGURE 3.1. $\Omega_{i,j}$.

The existence of such compact discs follows from a continuity argument. In order to achieve properties (E.3) and (E.4) take into account Claim 3.2.

Let $\eta : \{1, \dots, em\} \rightarrow \{1, \dots, e\} \times \mathbb{Z}_m$ be the bijection $\eta(k) = (E(\frac{k-1}{m}) + 1, k-1)$, where $E(\cdot)$ means integer part.

The second deformation process in the proof of Lemma 3.1 mainly acts in $\cup_k K_{\eta(k)}$ and is contained in the following claim.

Claim 3.3. *There exists a sequence $\{H_0 = H, H_1, \dots, H_{em}\} \subset \mathcal{N}(M)$ satisfying the following properties:*

- (F.1_k) $\|H_k - X\| < \epsilon_0$ on $\overline{M - ((\cup_{a=1}^k \Omega_{\eta(a)}) \cup (\cup_{a=1}^{em} K_{\eta(a)}))}$,
- (F.2_k) $\ll H_k - H_{k-1}, e_{\eta(k)} \gg = 0$,
- (F.3_k) $\|H_k(Q) - X(Q)\| > 2\rho + 1$, $\forall Q \in K_{\eta(a)}$, $\forall a \in \{1, \dots, k\}$, $k \geq 1$,

(F.4_k) if $J \subset \tilde{r}_{\eta(a)}$ is an arc connecting $\tilde{\alpha}_{\eta(a)} \cup \tilde{\alpha}_{\eta(a)+(0,1)}$ and $\tilde{\beta}_{\eta(a)} \cup \tilde{\beta}_{\eta(a)+(0,1)}$, $J_1 = J \cap \Omega_{\eta(a)}$ and $J_2 = J \cap \Omega_{\eta(a)+(0,1)}$, then

$$\mathcal{L}_{\mathbb{C}}((\mathcal{A}_{\eta(a)} \cdot H_k|_{J_1})_3) + \mathcal{L}_{\mathbb{C}}((\mathcal{A}_{\eta(a)+(0,1)} \cdot H_k|_{J_2})_3) > 2\rho \cdot \max\{\|\mathcal{A}_{\eta(a)}\|, \|\mathcal{A}_{\eta(a)+(0,1)}\|\},$$

$$\forall a \in \{1, \dots, \text{em}\},$$

(F.5_k) $\|H_k - H_{k-1}\| < \epsilon_0/\text{em}$ on $\overline{M - \Omega_{\eta(k)}}$, and

(F.6_k) $\|H_k - X\|_1 < \epsilon_0$ on M_0 .

Proof. Properties (F.1₀), (F.4₀) and (F.6₀) follow from (E.3), (E.4) and (D.3), whereas (F.2₀), (F.3₀) and (F.5₀) make no sense. We reason by induction. Assume that we have defined H_0, \dots, H_{k-1} satisfying the desired properties and let us construct H_k .

Label $G = \mathcal{A}_{\eta(k)} \cdot H_{k-1} \in \mathcal{N}(M)$ and write $G = (G_1, G_2, G_3)^T$.

Let γ denote a Jordan arc in $\tilde{\alpha}_{\eta(k)}$ disjoint from $\tilde{r}_{\eta(k)-(0,1)} \cup \tilde{r}_{\eta(k)}$, with initial point $\gamma_0 \in \alpha_{\eta(k)}$ and final point $\gamma_1 \in \partial K_{\eta(k)}$ and otherwise disjoint from $\partial \tilde{\alpha}_{\eta(k)}$. Choose γ so that $dG_3|_{\gamma}$ never vanishes. Denote $S_k = (\overline{M - \Omega_{\eta(k)}}) \cup \gamma \cup K_{\eta(k)}$, and without loss of generality assume that S_k is admissible.

Let $\gamma(u)$, $u \in [0, 1]$, be a smooth parameterization of γ with $\gamma(0) = \gamma_0$. Label $\tau_j = \gamma([0, 1/j])$ and consider the parameterization $\tau_j(u) = \gamma(u/j)$, $u \in [0, 1]$. Write $G_{3,j}(u) = G_3(\tau_j(u))$, $u \in [0, 1]$, and notice that $\frac{dG_{3,j}}{du}(0) = \frac{1}{j} \frac{d(G_3 \circ \gamma)}{du}(0)$ for all $j \in \mathbb{N}$.

From the definition of $\mathcal{A}_{\eta(k)}$, one has $\mathcal{A}_{\eta(k)}(\ll e_{\eta(k)} \gg^\perp) = \{(z_1, z_2, 0) \in \mathbb{C}^3 \mid z_1, z_2 \in \mathbb{C}\}$. Set $\zeta^\pm = \lambda(1, \pm i, 0) \in \mathbb{C}^3$, where $\lambda \in \mathbb{R}$,

$$(3.9) \quad \lambda > (2\rho + 2 + \epsilon_0)\|\mathcal{A}_{\eta(k)}\|,$$

and notice that $\zeta^\pm = \mathcal{A}_{\eta(k)}(\lambda(u_{\eta(k)} \pm v_{\eta(k)}))$. Set

$$\zeta_j = \zeta^+ - \frac{(dG_{3,j}/du(0))^2}{2 \prec \zeta^+, \zeta^- \succ} \zeta^- \in \mathcal{A}_{\eta(k)}(\ll e_{\eta(k)} \gg^\perp), \quad j \in \mathbb{N},$$

and observe that $\lim_{j \rightarrow \infty} \zeta_j = \zeta^+$ and $\prec \zeta_j, \zeta_j \succ = -(\frac{dG_{3,j}}{du}(0))^2$ for all j . Define $h_j : [0, 1] \rightarrow \mathbb{C}^3$ as

$$h_j(u) = G(\gamma_0) + i \frac{G_{3,j}(u) - G_{3,j}(0)}{\prec \zeta_j, \zeta_j \succ^{1/2}} \zeta_j + (0, 0, G_{3,j}(u) - G_3(\gamma_0)),$$

and notice that $\prec h'_j(u), h'_j(u) \succ = 0$ and $\ll h'_j(u), h'_j(u) \gg$ never vanishes on $[0, 1]$, $j \in \mathbb{N}$. Up to choosing a suitable branch of $\prec \zeta_j, \zeta_j \succ^{1/2}$, the sequence $\{h_j\}_{j \in \mathbb{N}}$ converges uniformly on $[0, 1]$ to $h_\infty : [0, 1] \rightarrow \mathbb{C}^3$, $h_\infty(u) = u\zeta^+ + G(\gamma_0)$. From equation (3.9) one has $\|h_\infty(1) - G(\gamma_0)\| = \|\zeta^+\| > (2\rho + 2 + \epsilon_0)\|\mathcal{A}_{\eta(k)}\|$, hence there exists a large enough $j_0 \in \mathbb{N}$ so that $\|G(\gamma_0) - G(\gamma(1/j_0))\| = \|G(\gamma_0) - G(\tau_{j_0}(1))\| < \|\mathcal{A}_{\eta(k)}\|$ and $\|h_{j_0}(1) - G(\gamma_0)\| > (2\rho + 2 + \epsilon_0)\|\mathcal{A}_{\eta(k)}\|$. In particular,

$$(3.10) \quad \|h_{j_0}(1) - G(\tau_{j_0}(1))\| > (2\rho + 1 + \epsilon_0)\|\mathcal{A}_{\eta(k)}\|.$$

Set $\hat{h} : \tau_{j_0} \rightarrow \mathbb{C}^3$, $\hat{h}(P) = h_{j_0}(u(P))$, where $u(P) \in [0, 1]$ is the only value for which $\tau_{j_0}(u(P)) = P$. Let $\hat{G} = (\hat{G}_1, \hat{G}_2, \hat{G}_3) : S_k \rightarrow \mathbb{C}^3$ denote the continuous map given by:

$$(3.11) \quad \hat{G}|_{\overline{M - \Omega_{\eta(k)}}} = G|_{\overline{M - \Omega_{\eta(k)}}}, \quad \hat{G}|_{\tau_{j_0}} = \hat{h}, \quad \hat{G}|_{(\gamma - \tau_{j_0}) \cup K_{\eta(k)}} = G|_{(\gamma - \tau_{j_0}) \cup K_{\eta(k)}} - G(\tau_{j_0}(1)) + \hat{h}(\tau_{j_0}(1)).$$

Notice that $\hat{G}_3 = G_3|_{S_k}$. The equation $\prec d\hat{G}, d\hat{G} \succ = 0$ formally holds except at the points γ_0 and $\tau_{j_0}(1)$ where smoothness could fail. Then, up to smooth approximation, (3.10) and (3.11) give that \hat{G} is a generalized null curve satisfying that

$$\hat{G}|_{\overline{M - \Omega_{\eta(k)}}} = G|_{\overline{M - \Omega_{\eta(k)}}}, \quad \hat{G}_3 = G_3|_{S_k}, \quad \text{and} \quad \|(\hat{G} - G)(Q)\| > (2\rho + 1 + \epsilon_0)\|\mathcal{A}_{\eta(k)}\| \quad \forall Q \in K_{\eta(k)}.$$

Fix $\epsilon_1 > 0$ which will be specified later. Applying Lemma 2.12 to S_k , M and \hat{G} , we get a null curve $Z = (Z_1, Z_2, Z_3)^T \in \mathcal{N}(M)$ such that

- $\|Z - \mathcal{A}_{\eta(k)} \cdot H_{k-1}\|_1 < \epsilon_1$ on $\overline{M - \Omega_{\eta(k)}}$,
- $Z_3 = (\mathcal{A}_{\eta(k)} \cdot H_{k-1})_3$, and
- $\|(Z - G)(Q)\| > (2\rho + 1 + \epsilon_0) \cdot \|\mathcal{A}_{\eta(k)}\| \forall Q \in K_{\eta(k)}$.

Define $H_k := \mathcal{A}_{\eta(k)}^{-1} \cdot Z \in \mathcal{N}(M)$. From the properties of Z above, H_k satisfies that

- (i) $\|H_k - H_{k-1}\|_1 < \epsilon_1 \|\mathcal{A}_{\eta(k)}^{-1}\|$ on $\overline{M - \Omega_{\eta(k)}}$,
- (ii) $\ll H_k - H_{k-1}, e_{\eta(k)} \gg = 0$ (see (3.4) and the definition of $\mathcal{A}_{\eta(k)}$), and
- (iii) $\|(H_k - H_{k-1})(Q)\| > (2\rho + 1 + \epsilon_0) \forall Q \in K_{\eta(k)}$.

Let us check that H_k is the null curve we are looking for, provided that ϵ_1 is sufficiently small. Indeed, (ii) directly gives (F.2_k), and if ϵ_1 is small enough, then (F.1_{k-1}), (F.6_{k-1}) and (i) guarantee (F.1_k), (F.5_k) and (F.6_k), and properties (F.1_{k-1}), (F.3_{k-1}), (i) and (iii) gives (F.3_k). Finally, to prove (F.4_k) observe that (i) gives that $\|\mathcal{A}_{\eta(a)} \cdot H_k - \mathcal{A}_{\eta(a)} \cdot H_{k-1}\| < \epsilon_1 \|\mathcal{A}_{\eta(k)}^{-1}\| \|\mathcal{A}_{\eta(a)}\|$ and $\|\mathcal{A}_{\eta(a)+(0,1)} \cdot H_k - \mathcal{A}_{\eta(a)+(0,1)} \cdot H_{k-1}\| < \epsilon_1 \|\mathcal{A}_{\eta(k)}^{-1}\| \|\mathcal{A}_{\eta(a)+(0,1)}\|$ on $\Omega_{\eta(a)} \forall a \neq k$, whereas (ii) gives that $(\mathcal{A}_{\eta(k)} \cdot H_k)_3 = (\mathcal{A}_{\eta(k)} \cdot H_{k-1})_3$ on $\Omega_{\eta(k)}$. Then (F.4_{k-1}) implies (F.4_k) provided that ϵ_1 is small enough. \square

Consider the null curve $\hat{X} := H_{\text{em}} \in \mathcal{N}(M)$. One has

Claim 3.4. $\text{dist}_{(M, \hat{X})}(M_0, \partial M) > 2\rho$. Moreover, $\text{dist}_{(M, \hat{X})}(M_0, \cup_{a=1}^{\text{em}} K_{\eta(a)}) > 2\rho$.

Proof. Indeed, consider a connected curve γ in M with initial point $Q \in M_0$ and final point $P \in \partial M$. Let us distinguish cases. Assume there exists $k \in \{1, \dots, \text{em}\}$ and a point $P_0 \in \gamma \cap K_{\eta(k)} \neq \emptyset$. Then there exists a point $Q_0 \in \gamma \cap (r_{\eta(k)-(0,1)} \cup \alpha_{\eta(k)} \cup r_{\eta(k)})$ and one has

$$\begin{aligned} \mathcal{L}_{(M, \hat{X})}(\gamma) = \mathcal{L}_{\mathbb{C}^3}(\hat{X}(\gamma)) &\geq \|\hat{X}(P_0) - \hat{X}(Q_0)\| \\ &\geq \|(\hat{X} - X)(P_0)\| - \|X(P_0) - X(Q_0)\| - \|(\hat{X} - X)(Q_0)\| \\ &\geq 2\rho + 1 - \epsilon_0 - \epsilon_0 > 2\rho, \end{aligned}$$

where $\mathcal{L}_{(M, \hat{X})}(\cdot)$ means length in M with respect to the metric $\hat{X}^* \ll \cdot, \cdot \gg$, $\mathcal{L}_{\mathbb{C}^3}(\cdot)$ means Euclidean length in \mathbb{C}^3 . To achieve these bounds we have used (F.3_{em}); (A.2) and (3.8); and (F.1_{em}); respectively (assume from the beginning $\epsilon_0 < 1/2$). This also proves the second part of the claim.

Assume now that $\gamma \cap (\cup_{k=1}^{\text{em}} K_{\eta(k)}) = \emptyset$. Then there exist $k \in \{1, \dots, \text{em}\}$ and a connected subarc $\hat{\gamma} \subset \gamma$ such that $P \in \beta_{\eta(k)} \cup \beta_{\eta(k)+(0,1)}$, $\hat{\gamma} \subset \tilde{r}_{\eta(k)}$ and $\hat{\gamma} \cap (\tilde{\alpha}_{\eta(k)} \cup \tilde{\alpha}_{\eta(k)+(0,1)}) \neq \emptyset$. Remark 2.9 and (F.4_{em}) give $\mathcal{L}_{(M, \hat{X})}(\gamma) \geq \mathcal{L}_{\mathbb{C}^3}(\hat{X}(\gamma)) \geq \mathcal{L}_{\mathbb{C}^3}(\hat{X}(\hat{\gamma})) > 2\rho$ and the proof is done. \square

Claim 3.5. $\|\hat{X}(Q) - \mathfrak{F}(\mathfrak{P}(Q))\| < \sqrt{4\rho^2 + \mu^2} + \epsilon$ for any $Q \in M - M_0$ with $\text{dist}_{(M, \hat{X})}(M_0, Q) < 2\rho$.

Proof. Obviously there exist $k \in \{1, \dots, \text{em}\}$ and $P \in r_{\eta(k)-(0,1)} \cup \alpha_{\eta(k)} \cup r_{\eta(k)}$ such that

$$(3.12) \quad Q \in \Omega_{\eta(k)} - K_{\eta(k)} \quad \text{and} \quad \text{dist}_{(M, \hat{X})}(P, Q) < 2\rho,$$

see Claim 3.4. By Pitagoras Theorem,

$$(3.13) \quad \|\hat{X}(Q) - \mathfrak{F}(\mathfrak{P}(Q))\| = \sqrt{|\ll \hat{X}(Q) - \mathfrak{F}(\mathfrak{P}(Q)), e_{\eta(k)} \gg|^2 + \|\Pi_{\eta(k)}(\hat{X}(Q) - \mathfrak{F}(\mathfrak{P}(Q)))\|^2}.$$

Let us bound the two addends in the right part of this inequality. On the one hand,

$$(3.14) \quad \begin{aligned} |\llbracket \hat{X}(Q) - \mathfrak{F}(\mathfrak{P}(Q)), e_{\eta(k)} \rrbracket| &\leq \|\hat{X}(Q) - H_k(Q)\| + |\llbracket (H_k - H_{k-1})(Q), e_{\eta(k)} \rrbracket| \\ &\quad + \|H_{k-1}(Q) - X(Q)\| + \|X(Q) - \mathfrak{F}(\mathfrak{P}(Q))\| \\ &< \epsilon_0 + 0 + \epsilon_0 + \mu, \end{aligned}$$

where we have used (F.5_a), $a = k + 1, \dots, \text{em}$; (F.2_k); (F.1_{k-1}); and (A.3). On the other hand,

$$(3.15) \quad \begin{aligned} \|\Pi_{\eta(k)}(\hat{X}(Q) - \mathfrak{F}(\mathfrak{P}(Q)))\| &\leq \|\hat{X}(Q) - \hat{X}(P)\| + \|\hat{X}(P) - X(P)\| + \|X(P) - X(P_{\eta(k)})\| \\ &\quad + \|\Pi_{\eta(k)}(X(P_{\eta(k)}) - \mathfrak{F}(\mathfrak{P}(P_{\eta(k)})))\| + \|\mathfrak{F}(\mathfrak{P}(P_{\eta(k)})) - \mathfrak{F}(\mathfrak{P}(Q))\| \\ &< 2\rho + \epsilon_0 + \epsilon_0 + \epsilon_0 + \epsilon_0, \end{aligned}$$

where we have used (3.12); (F.1_{em}); (A.2); (3.3); and (A.2); to get the corresponding bounds. Combining (3.13), (3.14) and (3.15) one has

$$\|\hat{X}(Q) - \mathfrak{F}(\mathfrak{P}(Q))\| < \sqrt{4\rho^2 + \mu^2 + (4\mu + 16\rho)\epsilon_0 + 20\epsilon_0} < \sqrt{4\rho^2 + \mu^2} + \epsilon,$$

provided that ϵ_0 is chosen from the beginning to satisfy the second inequality, and we are done. \square

By Claim 3.4 there exists $\mathcal{M} \in \mathcal{B}(\mathcal{R})$ such that

$$(G.1) \quad M_0 < \mathcal{M} < M, \text{ and}$$

$$(G.2) \quad \rho < \text{dist}_{(\mathcal{M}, \hat{X})}(M_0, Q) < 2\rho \quad \forall Q \in \partial\mathcal{M}.$$

Furthermore, up to infinitesimal variations of the boundary curves of \mathcal{M} , we can guarantee that

$$(G.3) \quad \hat{X}|_{\partial\mathcal{M}} : \partial\mathcal{M} \rightarrow \mathbb{C}^3 \text{ is an embedding.}$$

Set $\mathcal{X} := \hat{X}|_{\mathcal{M}} \in \mathcal{N}(\mathcal{M})$. Item (3.1.a) follows from (3.2) and (G.1). (G.3) directly gives (3.1.b). (3.1.c) is implied by (G.2). (3.1.d) follows from (F.6_{em}) and (3.2) provided that ϵ_0 is chosen less than ϵ . Finally, Claim 3.5 and (G.2) give (3.1.e). This proves Lemma 3.1.

4. Main Lemma

In this section we show the following density type result for null curves:

Lemma 4.1. *Let $V, W \in \mathcal{B}(\mathcal{R})$ with $V < W$. Let $Y \in \mathcal{N}(W)$ and let $\lambda > 0$.*

Then there exist $\mathcal{W} \in \mathcal{B}(\mathcal{R})$ and $\mathcal{Y} \in \mathcal{N}(\mathcal{W})$ satisfying that

$$(4.1.a) \quad \mathcal{Y}|_{\partial\mathcal{W}} : \partial\mathcal{W} \rightarrow \mathbb{C}^3 \text{ is an embedding,}$$

$$(4.1.b) \quad V < \mathcal{W} < W,$$

$$(4.1.c) \quad \lambda < \text{dist}_{(\mathcal{W}, \mathcal{Y})}(V, \partial\mathcal{W}),$$

$$(4.1.d) \quad \|\mathcal{Y} - Y\| < 1/\lambda \text{ on } \mathcal{W},$$

$$(4.1.e) \quad \delta^H(\mathcal{Y}(\partial\mathcal{W}), Y(\partial W)) < 1/\lambda, \text{ and}$$

$$(4.1.f) \quad \|\mathcal{Y} - Y\|_1 < 1/\lambda \text{ on } V.$$

The most important points in this lemma are Items (4.1.c) and (4.1.d). They assert that the immersion \mathcal{Y} is very close to Y in whole its domain of definition and, however, its intrinsic radius is much larger. Therefore, as we will see in Theorem 5.1 below, Lemma 4.1 is a powerful tool for constructing complete bounded null curves in \mathbb{C}^3 .

4.1. Proof of Lemma 4.1

Fix a positive constant ρ_1 which will be specified later. Consider sequences $\{\rho_n\}_{n \in \mathbb{N}}$ and $\{\mu_n\}_{n \in \mathbb{N}}$ given by

$$(4.1) \quad \rho_n = \rho_1 + \sum_{j=2}^n \frac{c}{j} \quad \text{and} \quad \mu_n = \sqrt{\mu_{n-1}^2 + 4\left(\frac{c}{n}\right)^2} + \frac{c}{n^2}, \quad \forall n \geq 2,$$

where c and μ_1 are small enough positive constants so that

$$(4.2) \quad \mu_n < \frac{1}{2\lambda} \quad \forall n \in \mathbb{N}.$$

For convenience write W_0 for W . Let T_0 be a metric tubular neighborhood of ∂W_0 in \mathcal{R} disjoint from \overline{V} and denote by $\mathfrak{P}_0 : T_0 \rightarrow \partial W_0$ the natural projection.

Claim 4.2. *There exists a sequence $\{\Xi_n = (W_n, T_n, Y_n)\}_{n \in \mathbb{N}}$, where $W_n \in \mathcal{B}(\mathcal{R})$, T_n is a metric tubular neighborhood of ∂W_n in \mathcal{R} and $Y_n \in \mathcal{N}(W_n)$, satisfying the following properties:*

- (1_n) $Y_n|_{\partial W_n} : \partial W_n \rightarrow \mathbb{C}^3$ is an embedding,
- (2_n) $V < W_n < W_{n-1} < W_0$ and $T_n \subset T_{n-1} \subset T_0$, $n \geq 1$,
- (3_n) $\rho_n < \text{dist}_{(W_n, Y_n)}(V, \partial W_n)$,
- (4_n) $\max\{\|Y - Y \circ \mathfrak{P}_0 \circ \dots \circ \mathfrak{P}_{n-1}\|, \|Y_n - Y \circ \mathfrak{P}_0 \circ \dots \circ \mathfrak{P}_{n-1}\|\} < \mu_n$ on ∂W_n , where $\mathfrak{P}_j : T_j \rightarrow \partial W_j$ is the corresponding orthogonal projection for all j , and
- (5_n) $\|Y_n - Y\|_1 < 1/\lambda$ on V .

Proof. Let us follow an inductive method. Take $W_1 \in \mathcal{B}(\mathcal{R})$, $V < W_1 < W_0$, close enough to W_0 so that $\partial W_1 \subset T_0$ and

$$(4.3) \quad \|Y - Y \circ \mathfrak{P}_0\| < \mu_1 \quad \text{on } \overline{W_0 - W_1}.$$

For the basis of the induction choose W_1 and $Y_1 = Y|_{W_1}$. This gives (4₁). (5₁) trivially holds. Assume that ρ_1 is small enough so that (3₁) is satisfied. Finally, up to an infinitesimal variation of ∂W_1 we can guarantee that (1₁) holds as well. Then choose T_1 any metric tubular neighborhood of ∂W_1 such that $T_1 \subset T_0$ and set $\Xi_1 = (W_1, T_1, Y_1)$.

For the inductive step, assume that Ξ_1, \dots, Ξ_{n-1} have been already constructed satisfying the desired properties, $n \geq 2$. Take $W'_n \in \mathcal{B}(\mathcal{R})$, $V < W'_n < W_{n-1}$, close enough to W_{n-1} so that $\partial W'_n \subset T_{n-1}$ and

$$(4.4) \quad \|Y - Y \circ \mathfrak{P}_0 \circ \dots \circ \mathfrak{P}_{n-1}\| < \mu_n \quad \text{on } \overline{W_{n-1} - W'_n}.$$

Here we have taken into account that $\mathfrak{P}_{n-1}|_{\partial W_{n-1}}$ is the identity map, (4_{n-1}) and the inequality $\mu_{n-1} < \mu_n$. Then set (W_n, Y_n) as the couple $(\mathcal{M}, \mathcal{X})$ obtained by Lemma 3.1 applied to the data

$$M = W_{n-1}, \quad T_{n-1}, \quad \mathfrak{P} = \mathfrak{P}_{n-1}, \quad \mu = \mu_{n-1}, \quad X = Y_{n-1},$$

$$\mathfrak{F} = (Y \circ \mathfrak{P}_0 \circ \dots \circ \mathfrak{P}_{n-2})|_{\partial W_{n-1}}, \quad N = W'_n, \quad \rho = \frac{c}{n} \quad \text{and} \quad \epsilon < \frac{c}{n^2},$$

and choose T_n any metric tubular neighborhood of ∂W_n such that $T_n \subset T_{n-1}$. Then (3.1.b), (3.1.a) and (3.1.c) give (1_n), (2_n) and (3_n), respectively. (4_n) follows from (3.1.e) and (4.4) (see also (4.1)), and finally (3.1.d) and (5_{n-1}) give (5_n) for ϵ small enough. \square

Since $\{\rho_n\}_{n \in \mathbb{N}} \nearrow +\infty$ then there exists $k \in \mathbb{N}$ such that $\rho_k > \lambda$. Define $\mathcal{W} = W_k$ and $\mathcal{Y} = Y_k$. Properties (4.1.a), (4.1.b), (4.1.c) and (4.1.f) trivially follow from (1_k), (2_k), (3_k) and (5_k), respectively. By (4_k) and (4.2) one has

$$\|\mathcal{Y} - Y\| \leq \|Y_k - Y \circ \mathfrak{P}_0 \circ \dots \circ \mathfrak{P}_{k-1}\| + \|Y \circ \mathfrak{P}_0 \circ \dots \circ \mathfrak{P}_{k-1} - Y\| < \mu_k + \mu_k < \frac{1}{\lambda} \quad \text{on } \partial \mathcal{W}.$$

Then, by the Maximum Principle for harmonic maps $\|\mathcal{Y} - Y\| < 1/\lambda$ on \mathcal{W} proving (4.1.d). Finally, (4.1.e) follows from the same argument. The proof of Lemma 4.1 is done.

5. Main Theorem

We are now ready to state and prove the main result of this paper. Observe that Main Theorem in the introduction directly follows from the following

Theorem 5.1. *Let $M, N \in \mathcal{B}(\mathcal{R})$, $N < M$. Let $X \in \mathcal{N}(M)$ and let $\epsilon > 0$.*

Then there exist a relatively compact domain $\mathcal{D} \subset \mathcal{R}$ and a compact complete null curve $\mathcal{X} : \overline{\mathcal{D}} \rightarrow \mathbb{C}^3$ satisfying that

- (5.1.a) $\mathcal{X}|_{Fr(\mathcal{D})} : Fr(\mathcal{D}) \rightarrow \mathbb{C}^3$ is an embedding,
- (5.1.b) $N \subset \mathcal{D} \subset \overline{\mathcal{D}} \subset M^\circ$, \mathcal{D} is homeomorphic to \mathcal{R} and both \mathcal{D} and $\overline{\mathcal{D}}$ are Runge,
- (5.1.c) $\|\mathcal{X} - X\|_1 < \epsilon$ on N and $\|\mathcal{X} - X\| < \epsilon$ on $\overline{\mathcal{D}}$,
- (5.1.d) $\delta^H(\mathcal{X}(Fr(\mathcal{D})), X(\partial M)) < \epsilon$, and
- (5.1.e) the Hausdorff dimension of $\mathcal{X}(Fr(\mathcal{D}))$ is 1.

Through the proof of Theorem 5.1 we will need the following notation. For any $k \in \mathbb{N}$, any compact set $K \subset \mathcal{R}$ and any continuous injective map $f : K \rightarrow \mathbb{C}^3$, set

$$\Psi(K, f, k) = \frac{1}{2k^2} \cdot \inf \{ \|f(P) - f(Q)\| \mid P, Q \in K, \text{dist}_{(\mathcal{R}, \omega)}(P, Q) > \frac{1}{k} \},$$

where $\text{dist}_{(\mathcal{R}, \omega)}(\cdot, \cdot)$ means intrinsic distance with respect to the Riemannian metric ω (see Remark 2.1). Notice that $\Psi(K, f, k) > 0$. The operator Ψ will be useful in order to guarantee that $\mathcal{X}|_{Fr(\mathcal{D})} : Fr(\mathcal{D}) \rightarrow \mathbb{C}^3$ is injective.

5.1. Proof of Theorem 5.1

Let c be a positive constant which will be specified later. Let $\epsilon_1 > 0$ and let $M_1 \in \mathcal{B}(\mathcal{R})$ satisfying that

- (i) $N < M_1 < M$,
- (ii) $X|_{\partial M_1} : \partial M_1 \rightarrow \mathbb{C}^3$ is an embedding,
- (iii) $\delta^H(X(\partial M_1), X(\partial M)) < \epsilon_1$, and
- (iv) $\epsilon_1 < c/2$.

Such M_1 exists by a continuity argument (Items (i) and (iii)). For Item (ii), make an infinitesimal variation of ∂M_1 if necessary. Set $X_1 := X|_{M_1}$ and let us prove the following

Claim 5.2. *There exists a sequence $\{\Theta_n = (M_n, X_n, \mathcal{T}_n, \epsilon_n, \tau_n)\}_{n \in \mathbb{N}}$, where $M_n \in \mathcal{B}(\mathcal{R})$, $X_n \in \mathcal{N}(M_n)$, \mathcal{T}_n is a metric tubular neighborhood of ∂M_n in M_n , $0 < \epsilon_n < c/2^n$, and $\tau_n > 0$, satisfying that*

- (1_n) $X_n|_{\overline{\mathcal{T}}_n} : \overline{\mathcal{T}}_n \rightarrow \mathbb{C}^3$ is an embedding,
- (2_n) $N < M_{n-1} - \mathcal{T}_{n-1} < M_n - \mathcal{T}_n < M_n < M_{n-1} < M$, $n \geq 2$,
- (3_n) $\|X_n - X_{n-1}\|_1 < \epsilon_n$ on $M_{n-1} - \mathcal{T}_{n-1}$ and $\|X_n - X_{n-1}\| < \epsilon_n$ on M_n , $n \geq 2$,
- (4_n) $\text{dist}_{(M_n, X_n)}(N, \mathcal{T}_n) > 1/\epsilon_n$, $n \geq 2$,
- (5_n) $\delta^H(X_n(\partial M_n), X_{n-1}(\partial M_{n-1})) < \epsilon_n$, $n \geq 2$,
- (6_n) there exist $a_n := e \cdot E((\tau_n)^{n+1})$ points $x_{n,1}, \dots, x_{n,a_n}$ in $X_n(\partial M_n) \subset \mathbb{C}^3$ such that

$$\text{dist}_{\mathbb{C}^3}(X_n(\overline{\mathcal{T}}_n), \{x_{n,1}, \dots, x_{n,a_n}\}) < (1/\tau_n)^n,$$

where $E(\cdot)$ and $\text{dist}_{\mathbb{C}^3}(\cdot)$ means integer part and Euclidean distance in \mathbb{C}^3 , respectively,

- (7_n) $\epsilon_n < \min \left\{ \epsilon_{n-1}, \varrho_{n-1}, \frac{1}{n^2(\tau_{n-1})^n}, \Psi(\overline{\mathcal{T}}_{n-1}, X_{n-1}|_{\overline{\mathcal{T}}_{n-1}}, n) \right\}$, where

$$\varrho_{n-1} = 2^{-n} \min \left\{ \min_{M_{k-1} - \mathcal{T}_{k-1}} \left\| \frac{dX_k}{\omega} \right\| \mid k = 1, \dots, n-1 \right\} > 0, \quad n \geq 2,$$

- (8_n) $\tau_n \geq \tau_{n-1} + 1 \geq n$, $n \geq 2$, and
 (9_n) $\max\{\text{dist}_{(\mathcal{R}, \omega)}(P, \partial M_n) \mid P \in \overline{T}_n\} < \epsilon_n$.

Proof. Let us follow an inductive process. We have already introduced M_1 , X_1 and ϵ_1 , hence for setting Θ_1 we only must define $T(M_1)$ and τ_1 satisfying the corresponding properties.

Set $\tau_1 = \max\{1, \mathcal{L}_{\mathbb{C}^3}(X_1(\partial M_1))\}$, where $\mathcal{L}_{\mathbb{C}^3}(\cdot)$ means Euclidean length of curves in \mathbb{C}^3 .

Write $\gamma_1, \dots, \gamma_e$ for the components of ∂M_1 . For any $i = 1, \dots, e$, let $y_{i,1}, \dots, y_{i,a_1/e}$ be points on $X_1(\gamma_i)$ with mutual distance along $X_1(\gamma_i)$ and denote by $b_{1,i}$ this distance. Set $b_1 = \max\{b_{1,1}, \dots, b_{1,e}\}$ and $\{x_{1,1}, \dots, x_{1,a_1}\} = \{y_{i,j} \mid i = 1, \dots, e, j = 1, \dots, a_1/e\}$. Then $b_1 < 1/\tau_1$, hence

$$(5.1) \quad \text{dist}_{\mathbb{C}^3}(X_1(\partial M_1), \{x_{1,1}, \dots, x_{1,a_1}\}) < 1/\tau_1.$$

By (i), (ii), (5.1) and a continuity argument, there exists a tubular neighborhood $T(M_1)$ of ∂M_1 on M_1 such that (1₁), (6₁), (9₁) and $N < M_1 - T(M_1)$ (which is the meaningful part of (2₁)) hold. The remaining properties of Θ_1 make no sense.

For the inductive step, assume that we have stated $\Theta_1, \dots, \Theta_{n-1}$, $n \geq 2$, satisfying the corresponding properties and let us construct Θ_n .

Observe that (1_{n-1}) implies that $\Psi(\overline{T}_{n-1}, X_{n-1}|_{\overline{T}_{n-1}}, n) > 0$. Then we can choose $\epsilon_n < c/2^n$ satisfying (7_n). Define $M_n = \mathcal{W} \in \mathcal{B}(\mathcal{R})$ and $X_n = \mathcal{Y} \in \mathcal{N}(\mathcal{W})$ given by Lemma 4.1 applied to the data

$$V = M_{n-1} - T_{n-1}, \quad W = M_{n-1}, \quad Y = X_{n-1} \quad \text{and} \quad \lambda = \frac{1}{\epsilon_n}.$$

Then one has

- (a) $X_n|_{\partial M_n} : \partial M_n \rightarrow \mathbb{C}^3$ is an embedding,
- (b) $M_{n-1} - T_{n-1} < M_n < M_{n-1}$,
- (c) $\text{dist}_{(M_n, X_n)}(M_{n-1} - T_{n-1}, \partial M_n) > 1/\epsilon_n$,
- (d) $\|X_n - X_{n-1}\|_1 < \epsilon_n$ on $M_{n-1} - T_{n-1}$ and $\|X_n - X_{n-1}\| < \epsilon_n$ on M_n , and
- (e) $\delta^H(X_n(\partial M_n), X_{n-1}(\partial M_{n-1})) < \epsilon_n$.

Items (d) and (e) directly give (3_n) and (5_n). Set $\tau_n = \max\{\tau_{n-1} + 1, \mathcal{L}_{\mathbb{C}^3}(X_n(\partial M_n))\}$. Hence (8_n) holds. Write ξ_1, \dots, ξ_e for the components of ∂M_n . Recall that $X_n(\xi_i)$ is a Jordan curve by (a), $i = 1, \dots, e$. For any $i = 1, \dots, e$ consider $z_{i,1}, \dots, z_{i,a_n/e}$ points in $X_n(\xi_i)$ with mutual distance along $X_n(\xi_i)$ and denote by $b_{n,i}$ this distance. Set $b_n = \max\{b_{n,i}\}_{i=1}^e$ and $\{x_{n,1}, \dots, x_{n,a_n}\} = \{z_{i,j} \mid i = 1, \dots, e, j = 1, \dots, a_n/e\}$. Then $b_n < 1/(\tau_n)^n$, hence

$$(5.2) \quad \text{dist}_{\mathbb{C}^3}(X_n(\partial M_n), \{x_{n,1}, \dots, x_{n,a_n}\}) < (1/\tau_n)^n.$$

Take a metric tubular neighborhood T_n of ∂M_n in M_n . By a continuity argument and choosing T_n small enough, all the properties from (1_n) to (9_n) hold. Indeed, (9_n) can be trivially guaranteed, and for (1_n); (2_n); (4_n); and (6_n), take into account (a); (2_{n-1}) and (b); (c); and (5.2), respectively. This proves the claim. \square

Set $\mathcal{N}_n = (M_n - T_n)^\circ$ and $\mathcal{S}_n = \mathcal{R} - M_n$ for all $n \in \mathbb{N}$. Define

$$\mathcal{D} := \bigcup_{n \in \mathbb{N}} \mathcal{N}_n$$

and notice that, by (2_n) and (9_n), $n \in \mathbb{N}$, one has

$$\overline{\mathcal{D}} = \bigcap_{n \in \mathbb{N}} M_n.$$

By (2_n), $n \in \mathbb{N}$, and elementary topological arguments one has that \mathcal{D} is Runge and homeomorphic to \mathcal{R} . On the other hand \mathcal{S}_n consists of e pairwise disjoint open annuli and $\mathcal{S}_n \subset \mathcal{S}_{n+1} \subset$

$\mathcal{R} - N \forall n \in \mathbb{N}$, hence $\mathcal{R} - \overline{\mathcal{D}} = \cup_{n \in \mathbb{N}} \mathcal{S}_n$ consists of pairwise disjoint open annuli as well. This implies that $\overline{\mathcal{D}}$ is a Runge compact set. Then Item (5.1.b) follows from (2_n), $n \in \mathbb{N}$.

By (2_n), (3_n) and (7_n), $n \in \mathbb{N}$, $\{X_n|_{\overline{\mathcal{D}}}\}_{n \in \mathbb{N}}$ uniformly converges to a continuous map

$$\mathcal{X} : \overline{\mathcal{D}} \rightarrow \mathbb{C}^3$$

with $\|\mathcal{X} - X\|_1 < c$ on N and $\|\mathcal{X} - X\| < c$ on $\overline{\mathcal{D}}$. This, (iii) and (iv) give (5.1.c) and (5.1.d) provided that $c < \epsilon$ from the beginning. By (3_n), $n \in \mathbb{N}$, $\mathcal{X}|_{\mathcal{D}}$ is a holomorphic map and $\langle d\mathcal{X}, d\mathcal{X} \rangle = 0$ on \mathcal{D} . To check that $\langle d\mathcal{X}, d\mathcal{X} \rangle$ never vanishes on \mathcal{D} argue as follows. Fix $P \in \mathcal{D}$ and take $n_0 \in \mathbb{N}$ such that $P \in M_{n-1} - \mathcal{T}_{n-1} \forall n \geq n_0$. Then

$$\begin{aligned} \left\| \frac{d\mathcal{X}}{\omega} \right\|(P) &\geq \left\| \frac{dX_{n_0}}{\omega} \right\|(P) - \sum_{n > n_0} \|X_n - X_{n-1}\|_1 \\ &\geq \left\| \frac{dX_{n_0}}{\omega} \right\|(P) - \sum_{n > n_0} \epsilon_n \\ &\geq \left\| \frac{dX_{n_0}}{\omega} \right\|(P) - \sum_{n > n_0} \varrho_{n-1} \\ &\geq \left(1 - \sum_{n > n_0} 2^{-n}\right) \left\| \frac{dX_{n_0}}{\omega} \right\|(P), \end{aligned}$$

where we have used (3_n) and (7_n). Therefore

$$(5.3) \quad \left\| \frac{d\mathcal{X}}{\omega} \right\|(P) \geq \frac{1}{2} \left\| \frac{dX_{n_0}}{\omega} \right\|(P) > 0,$$

In particular $\langle d\mathcal{X}, d\mathcal{X} \rangle(p) \neq 0$ and $\mathcal{X}|_{\mathcal{D}}$ is an immersion. This proves that $\mathcal{X}|_{\mathcal{D}}$ is a null curve.

From (4_n), $n \in \mathbb{N}$, and (5.3), one has

$$\text{dist}_{(\mathcal{D}, \mathcal{X})}(N, Fr(\mathcal{D})) = \lim_{n \rightarrow \infty} \text{dist}_{(\mathcal{D}, \mathcal{X})}(N, \mathcal{T}_n) \geq \frac{1}{2} \lim_{n \rightarrow \infty} \text{dist}_{(M_n, X_n)}(N, \mathcal{T}_n) > \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{\epsilon_n} = \infty,$$

hence $\mathcal{X}|_{\mathcal{D}}$ is complete and $\mathcal{X} : \overline{\mathcal{D}} \rightarrow \mathbb{C}^3$ is a compact complete null curve as claimed.

Let us show that $\mathcal{X}|_{Fr(\mathcal{D})} : Fr(\mathcal{D}) \rightarrow \mathbb{C}^3$ is injective. Indeed, take points P and Q in $Fr(\mathcal{D}) \subset \mathcal{T}_n$ (see (2_n)), $n \in \mathbb{N}$, $P \neq Q$. Take $n_0 \in \mathbb{N}$ large enough so that $\text{dist}_{(\mathcal{R}, \omega)}(P, Q) > 1/n_0$. Then, for any $n > n_0$,

$$\begin{aligned} \|X_{n-1}(P) - X_{n-1}(Q)\| &\leq \|X_{n-1}(P) - X_n(P)\| + \|X_n(P) - X_n(Q)\| + \|X_n(Q) - X_{n-1}(Q)\| \\ &< \epsilon_n + \|X_n(P) - X_n(Q)\| + \epsilon_n \\ &< \frac{1}{n^2} \|X_{n-1}(P) - X_{n-1}(Q)\| + \|X_n(P) - X_n(Q)\|, \end{aligned}$$

where we have used (3_n), (7_n) and the definition of Ψ . Then

$$\|X_{n_0+k}(P) - X_{n_0+k}(Q)\| > \left(\prod_{n=n_0+1}^{n_0+k} \left(1 - \frac{1}{n^2}\right) \right) \|X_{n_0}(P) - X_{n_0}(Q)\| \quad \forall k \in \mathbb{N}.$$

Taking limits in this expression as k goes to infinity, $\|\mathcal{X}(P) - \mathcal{X}(Q)\| > \frac{1}{2} \|X_{n_0}(P) - X_{n_0}(Q)\| > 0$ and (5.1.a) holds.

Finally let us check (5.1.e). Take $n \in \mathbb{N}$ with $\sum_{m=n}^{\infty} 1/m^2 < 1$. Then, for any $P \in \text{Fr}(\mathcal{D})$ and any $k > n$, properties (7_m) and (9_m) , $m = n+1, \dots, k$, give

$$\begin{aligned} \|X_k(P) - X_n(P)\| &\leq \|X_k(P) - X_{k-1}(P)\| + \dots + \|X_{n+1}(P) - X_n(P)\| \\ &< \frac{1}{k^2(\tau_{k-1})^k} + \dots + \frac{1}{(n+1)^2(\tau_n)^{n+1}} \\ &< \left(\sum_{m=n+1}^k \frac{1}{m^2} \right) \frac{1}{(\tau_n)^{n+1}} < \frac{1}{(\tau_n)^{n+1}} < \frac{1}{(\tau_n)^n}. \end{aligned}$$

Taking limits in the above inequality as k goes to infinity one has $\|\mathcal{X}(P) - X_n(P)\| < 1/(\tau_n)^n$ for any large enough $n \in \mathbb{N}$. Combining this inequality and properties (6_n) one obtains that $\text{dist}(\mathcal{X}(\text{Fr}(\mathcal{D})), \{x_{n,1}, \dots, x_{n,a_n}\}) < 2/(\tau_n)^n$. Since $a_n \cdot (2/(\tau_n)^n)^{1+1/n} < 4e$ then the Hausdorff measure $\mathcal{H}^1(\mathcal{X}(\text{Fr}(\mathcal{D}))) < \infty$, and so the Hausdorff dimension of $\mathcal{X}(\text{Fr}(\mathcal{D}))$ is at most 1. On the other hand, if c is taken small enough from the beginning, then each of the e connected components of $\mathcal{X}(\text{Fr}(\mathcal{D}))$ contains more than one point (see Item (5.1.c)). Therefore the Hausdorff dimension of $\mathcal{X}(\text{Fr}(\mathcal{D}))$ is at least 1, hence equals to 1. This proves (5.1.e) and the theorem.

5.2. Applications to other ambient manifolds

Given an open Riemann surface \mathcal{N} , a map $\varphi : \mathcal{N} \rightarrow \text{SL}(2, \mathbb{C})$ is said to be a null curve if and only if φ is a holomorphic immersion and $\det(d\varphi) = 0$. The hyperbolic 3-space \mathbb{H}^3 can be identified to $\text{SL}(2, \mathbb{C})/\text{SU}(2)$, and Bryant's projection

$$\mathcal{B} : \text{SL}(2, \mathbb{C}) \rightarrow \mathbb{H}^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2), \quad \mathcal{B}(A) = A \cdot \overline{A}^T,$$

maps complete null curves in $\text{SL}(2, \mathbb{C})$ into conformal complete immersions of constant mean curvature 1 in \mathbb{H}^3 .

Our Main Theorem directly implies similar existence results for complex curves in \mathbb{C}^2 , minimal surfaces in \mathbb{R}^3 , null curves in $\text{SL}(2, \mathbb{C})$ and Bryant surfaces in \mathbb{H}^3 of arbitrary finite topology (see [MUY1, MUY2, AL2] and the references therein for more details). All of them have been compiled in the following corollary.

Corollary 5.3. *Let S be an open orientable surface of finite topology and $\mathfrak{M} \in \{\mathbb{C}^2, \mathbb{R}^3, \text{SL}(2, \mathbb{C}), \mathbb{H}^3\}$.*

Then there exist an open Riemann surface \mathcal{M} , a relatively compact domain $\mathcal{D} \subset \mathcal{M}$ such that \mathcal{D} is homeomorphic to S and both \mathcal{D} and $\overline{\mathcal{D}}$ are Runge subsets in \mathcal{M} , and a continuous map $X : \overline{\mathcal{D}} \rightarrow \mathfrak{M}$ satisfying that $X|_{\text{Fr}(\mathcal{D})} : \text{Fr}(\mathcal{D}) \rightarrow \mathfrak{M}$ is an embedding, the Hausdorff dimension of $X(\text{Fr}(\mathcal{D}))$ is 1 and either of the following statements:

- (i) $\mathfrak{M} = \mathbb{C}^2$ and $X|_{\mathcal{D}}$ is a complete holomorphic immersion. Furthermore, if $\epsilon > 0$ and Σ is a finite family of closed curves in \mathbb{C}^2 which is spanned by a compact complex curve in \mathbb{C}^2 which is homeomorphic to S and can be lifted to a null curve in \mathbb{C}^3 , then X and \mathcal{D} can be chosen so that $\delta_{\mathbb{C}^2}^H(\Sigma, X(\text{Fr}(\mathcal{D}))) < \epsilon$, where $\delta_{\mathbb{C}^2}^H(\cdot, \cdot)$ means Hausdorff distance in \mathbb{C}^2 .
- (ii) $\mathfrak{M} = \mathbb{R}^3$, $X|_{\mathcal{D}}$ is a conformal complete minimal immersion and the conjugate immersion $(X|_{\mathcal{D}})^*$ of $X|_{\mathcal{D}}$ is well defined, extends continuously to $\overline{\mathcal{D}}$ and satisfies the same properties. Furthermore, if $\epsilon > 0$ and Σ, Σ^* are two finite families of closed curves in \mathbb{R}^3 spanned by a minimal surface homeomorphic to S and its (well defined) conjugate surface, respectively, then X and \mathcal{D} can be chosen so that $\max\{\delta_{\mathbb{R}^3}^H(\Sigma, X(\text{Fr}(\mathcal{D}))), \delta_{\mathbb{R}^3}^H(\Sigma^*, X^*(\text{Fr}(\mathcal{D})))\} < \epsilon$, where $\delta_{\mathbb{R}^3}^H(\cdot, \cdot)$ means Hausdorff distance in \mathbb{R}^3 .
- (iii) $\mathfrak{M} = \text{SL}(2, \mathbb{C})$ and $X|_{\mathcal{D}}$ is a complete null holomorphic curve. Furthermore, if $\epsilon > 0$ and Σ is a finite family of closed curves in $\text{SL}(2, \mathbb{C})$ spanned by a null holomorphic curve homeomorphic to S , then X and \mathcal{D} can be chosen so that $\delta_{\text{SL}(2, \mathbb{C})}^H(\Sigma, X(\text{Fr}(\mathcal{D}))) < \epsilon$, where $\delta_{\text{SL}(2, \mathbb{C})}^H(\cdot, \cdot)$ means Hausdorff distance in $\text{SL}(2, \mathbb{C})$.

- (iv) $\mathfrak{M} = \mathbb{H}^3$ and $X|_{\mathcal{D}}$ is a conformal complete CMC-1 surface in \mathbb{H}^3 . Furthermore, if $\epsilon > 0$ and Σ is a finite family of closed curves in \mathbb{H}^3 spanned by a CMC-1 surface which is homeomorphic to S and can be lifted to a null holomorphic curve in $\mathrm{SL}(2, \mathbb{C})$, then X and \mathcal{D} can be chosen so that $\delta_{\mathbb{H}^3}^H(\Sigma, X(\mathrm{Fr}(\mathcal{D}))) < \epsilon$, where $\delta_{\mathbb{H}^3}^H(\cdot, \cdot)$ means Hausdorff distance in \mathbb{H}^3 .

The proofs of Items (i), (ii) and (iv) follow straightforwardly except for the fact that $X|_{\mathrm{Fr}(\mathcal{D})} : \mathrm{Fr}(\mathcal{D}) \rightarrow \mathfrak{M}$ is injective. However, this can be always guaranteed by trivial refinements of Lemmas 3.1 and 4.1 and Theorem 5.1. Item (ii) generalizes the existence results obtained by the first author in [A1].

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